Suppose you have a headache and take a painkiller such as aspirin, paracetamol or ibuprofen. The drug will be absorbed and taken by your bloodstream to all parts of your body. Some organs of your body, most notably the kidneys, will begin to remove the drug.

The rate at which this drug clearance takes place can be assumed to be proportional to the quantity of the drug in your body. In this activity you will use calculus to develop a mathematical model for the process.

Information sheet Defining the problem

Suppose that the amount of a drug in your body t minutes after it reaches its peak is m milligrams. Then the rate at which the drug reduces is given by:

$\frac{dm}{dt} = -km$ where k is a positive constant.

Think about

Can you explain this equation?

This differential equation can be solved by separating the variables and integrating

$$\int \frac{1}{m}\,dm = \int -k\,dt$$

$$\ln m = -kt + c \quad \ldots\ldots\ldots\ldots(1)$$

To find the value of the constant c and an expression for m

The constant c can be found by estimating the maximum amount of the drug in your body. Such an estimate can be found from urine tests. The usual maximum amount of paracetamol taken is 500 mg.

Substituting $m = 500$, $t = 0$ into (1) gives $\ln 500 = c$

and so from (1) $\ln m = -kt + \ln 500$

$\ln m - \ln 500 = -kt$

Think about

What are the laws of logarithms? Which one is useful here?

The laws of logarithms give $\ln \frac{m}{500} = -kt$ ……………………. (2)

$\frac{m}{500} = e^{-kt}$

$m = 500e^{-kt}$ ……………….. (3)

Think about

What was the advantage of leaving c as ln 500 rather than using its value of 6.215?

To find the value of the constant *k* more information is needed

The time taken for the amount of drug in the body to reduce to half its maximum value is called its **half-life**. The half-life differs from one drug to another. The half-life of paracetamol is about 2 hours or 120 minutes, at which time only 250 mg would remain in the body.

Substituting m = 250, t = 120 into (2) gives

ln 0.5 = – 120k $\quad\Rightarrow\quad k \;= 0.00578$

From (2) and (3), the relationship between m and t can be written as either

$$t = -173 \ln \frac{m}{500} \quad \text{or} \quad m = 500e^{-0.00578t}$$

These equations allow us to predict how long it will take the drug level to reduce to a particular value, and also how much of the drug will remain in your body after a particular time interval.

Your headache should disappear soon. If not, after a while you may need to take another dose of painkiller in order to keep the drug level in your body high enough to counteract the pain.

Think about

What is the significance of constant 500 in the equations for t and for m?

In the equation for t, will the constant in front of the ln term always be negative? Why?

Try these

1 A different painkiller has a half-life of 75 minutes. After taking this painkiller, the drug level in the patient reaches a peak value of 300 mg, and then reduces at a rate proportional to the amount of drug present in the body.

a Explain why the amount of drug in the body, m milligrams, at time t minutes after the drug level reaches its peak, is given by

$$\frac{\mathrm{d}m}{\mathrm{d}t} = -km \quad \text{where } k \text{ is a positive constant.}$$

b Solve this differential equation and show that the solution can be written in either of the forms:

$$t = -108 \ln \frac{m}{300} \qquad \text{or} \qquad m = 300e^{-0.00924t}$$

c For each of the drugs (those on the Information sheet and described above), calculate:

i the time taken for the drug level to reduce to 100 g

ii the amount of drug left in the body after 4 hours.

d i On the same axes, draw graphs of the functions

$m = 500e^{-0.0154t}$ and $m = 300e^{-0.00924t}$

ii Describe the main features of the graphs, their similarities and differences, and explain how these relate to the real situation.

2 Many popular drinks such as coffee, tea and cola contain caffeine.

The level of caffeine in the bloodstream reaches a peak between 15 and 45 minutes after the drink is consumed. Then it begins to fall (assuming that the person does not have another drink containing caffeine) at a rate proportional to the amount of caffeine left in the body.

The rate at which the caffeine is eliminated from the body varies from person to person.

The first table below gives estimates of the half-life of caffeine for different types of people. The second table gives estimates of the peak level of caffeine in the body after different drinks.

Person	Half-life
Non-smoking adult	6 hours
Smoker	4 hours
Woman using oral contraceptive pill	12 hours
Pregnant woman	18 hours
Child	3 hours

Drink	Peak caffeine level
Tea	70 mg
Filter coffee	150 mg
Instant coffee	90 mg
Cola	35 mg

a Choose a particular type of person and one of the drinks.
By solving a differential equation, find an equation that models the amount of caffeine, m milligrams, left in the body t hours after it reaches its peak.

b Investigate how the model varies from one drink to another and one type of person to another.

Reflection

The rate at which your body removes drugs is proportional to the quantity of the drug that remains in your body.

What is the differential equation that models this situation?

What is the typical form of the solution?

Can you sketch a typical graph?

What is meant by the half-life of a drug?

Is the half-life of a drug the same for all people?