Children's understanding of probability

A literature review (summary report)

Peter Bryant and Terezinha Nunes
University of Oxford

www.nuffieldfoundation.org

Nuffield Foundation
28 Bedford Square
London WC1B 3JS
Telephone: +44 (0)20 7631 0566
www.nuffieldfoundation.org

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This is a summary of a longer literature review which is available to download in full from www.nuffieldfoundation.org/probability.

Children's understanding of probability was commissioned by the Nuffield Foundation following the 2009 publication of a wider review into how children learn mathematics: *Key understandings in mathematics learning,* by Peter Bryant, Terezinha Nunes and Anne Watson. *Key understandings* is available to download from www.nuffieldfoundation.org/key-understandings.

Foreword

In 2009, the Nuffield Foundation published *Key understandings in mathematics learning,* a review of the research literature on how children learn mathematics. It has been widely read and has already had an impact on mathematics teaching and policy in several countries.

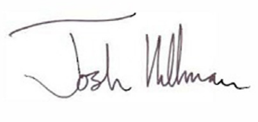
Probability was not included in *Key understandings,* and we subsequently commissioned two of its authors, Professors Peter Bryant and Terezinha Nunes from the University of Oxford, to examine the evidence on this topic. This report is a summary of their review, which is available to download in full on our website.

There are four key reasons for our interest in probability. First, we wondered whether the teaching and learning of probability took sufficient account of children's prior knowledge of fairness, randomness and chance - concepts which are acquired at a very young age and which lay the foundations for probabilistic thinking. *Key understandings* noted that primary school geometry often failed to build on children's pre-school knowledge of spatial relations, and we thought probability might offer an interesting parallel. Second, the extent to which probability forms part of the primary curriculum has been subject to change in recent decades and so a consolidation of the evidence is timely. Third, this evidence is essential to underpin further research and development work, and fourth, probability is particularly relevant to our interest in statistical literacy in the wider population. Adults as well as children often find it difficult to think rationally about probability and randomness, so early encounters with these concepts are important.

In this review, the authors identify four 'cognitive demands' made on children when learning about probability, and examine evidence in each of these areas: randomness, the sample space, comparing and quantifying probabilities, and correlations. They draw together international evidence, from the early years through to adulthood, and highlight studies that are of particular relevance to teaching. They also identify areas that have been relatively neglected and would benefit from further research, particularly from fully evaluated intervention projects.

Indeed, the authors are currently seeking to address some of these gaps through a large-scale controlled study of the teaching of probability to 9-to-10-year-olds, which the Foundation is pleased to be funding.

We are grateful to the authors for their unstinting enthusiasm and commitment to this topic. The review is an informative and engaging read for anyone interested in how we understand (and misunderstand) probability, and provides valuable evidence that could be used to inform both teaching approaches and the design of future research.



Josh Hillman, Director of Education

Summary of the review

The 'cognitive demands' of understanding probability

Many of the events and relations in people's lives are well understood and entirely predictable. If we knock a glass over, the liquid in it spills. If John is Michael's father, John must be older than Michael. Other events and associations, such as a road accident or winning a lottery, are less predictable because they happen randomly. People know they might happen, but are uncertain if and when they will happen.

We can, nevertheless, reason logically about random events. This reasoning allows us to work out the probability of particular outcomes, and thus to understand the risks and possible benefits of acting in one way rather than another.

The understanding of the implications of randomness also lies at the centre of all statistical thinking. We decide the significance of any difference, for example in the recovery rates of patients given a specific drug and of others given a placebo, by calculating whether this difference could have happened by chance. Many associations, such as the association between income and health, are imperfect, and the most effective way of working out whether there is a genuine relation between two variables is to work out how much of the association could be due to random factors.

Randomness and uncertainty play an important part in scientific thinking as well, since many physical processes, such as the movement of subatomic particles are random, and need to be analysed in terms of probability.

Another good reason for people to be able to think rationally about randomness and uncertainty is that randomisation plays an important part in ensuring fairness in their every daily lives. Playing cards are shuffled and people are selected by lot to ensure that no one is given an unfair start.

Despite the central importance of randomness and probability in our lives, it is clear that children, and many adults as well, often have great difficulty in thinking rationally about, and quantifying, probability. Probability is quite a complex concept, and in order to learn about it we have to draw on our understanding of four different aspects of events and the sequence in which they occur. These four 'cognitive demands', as we call them in the report, are:

- *Understanding randomness:* To understand the nature and the consequences of randomness, and the use of randomness in our everyday lives.

- *Working out the sample space:* To recognise that the first and essential step in solving any probability problem is to work out all the possible events and sequences of events that could happen. The set of all the possible events is called 'the sample space' and working out the sample space is not just a necessary part of the calculation of the probabilities of particular event, but also an essential element in understanding the nature of probability.

- *Comparing and quantifying probabilities:* Probabilities are quantities based on proportions, and one has to calculate these proportions to make most (but not all) comparisons of the probabilities of two or more events. These proportions can be expressed as decimals, as fractions or as ratios.

- *Understanding correlation (or relationships between events):* An association between two kinds of event could happen randomly or, alternatively, could represent a genuine relationship. To discover whether there is a non-random relation or not, we have to attend to the relation between confirming and disconfirming evidence and check whether the frequency of confirming cases could have happened by chance. This means that, in order to understand correlations, we need to understand all three ideas mentioned above.

Randomness

Randomisation is a common and important part of everyday life, but it is clear that many adults' grasp of the nature of randomness and its consequences is quite tenuous. Research on young children suggests they have even more difficulty understanding randomness than adults.

Some aspects of randomness may be easier to understand than others. There are claims, for example, that even babies can understand the link between uncertainty and randomness. One study apparently shows that babies realise that choices made by people who cannot see what they are doing will be random, and governed by probability, whereas people who can see what they are doing will choose items that they want (Denison and Xu, 2009). However, problems with the design of this study mean it is not possible to reach a definite conclusion about this.

Piaget and Inhelder (1975) were the first to study children's understanding of randomness. In a classic experiment, they progressively randomised the position of marbles of two different colours, which were initially grouped by colour at one end of a tray, by tilting the tray and letting the marbles roll to the other side, and then by tilting it back and forth repeatedly. Young children could not predict the consequent jumbling of the two colours. However, this context was probably strange to the children, and the study needs to be done again with forms of randomisation, like shuffling cards, that are more familiar to children.

Research, using computer microworlds, has shown that by the age of about ten, many children realise that there is an association between randomness and fairness, and that randomisation can be an effective way of ensuring fair allocations (Pratt and Noss, 2002; Paparistodemou et al, 2008; Watson et al, in press). This association could be used to teach children more about the nature of randomness.

A common mistake made by adults and children, is to disregard the independence of successive events in a random situation. One's chance of getting a tail on the next toss of a coin is not affected by what happened on previous throws. Even if the last six throws were all tails, the result is no more or less likely to be a tail again on the next throw than it was on the first. Many people make the mistake of judging that, after a run of one kind of outcome, a different outcome is more likely the next time round. This is called the 'negative recency' effect. Another kind of mistake, called the 'positive recency' effect, is to predict after a run of one outcome that the same outcome is more likely to happen the next time. Many adults (Gilovich et al, 1985) and most children make these mistakes, but recent research shows a higher proportion of positive recency errors among children than among adults and vice versa with negative recency errors (Chiesi and Primi, 2009).

Sample space

We can only calculate the probabilities of particular events if we know what all the possibilities are. The complete set of possibilities in a probability problem is called its 'sample space'. Working out the sample space is the essential first step in solving any probability problem (Keren, 1984; Chernoff, 2009), and in many it is the most important, since the solution is often quite obvious to someone who knows all the possibilities. Yet this aspect of probability has been relatively neglected in research on children's ideas about chance, which has concentrated for the most part on children's understanding of randomness and on their ability to quantify and compare probabilities.

Much of the information on people's awareness and use of sample space comes from mistakes that children and adults make in reasoning about probability, which they wouldn't have made if they had a thorough grasp of the relevant sample space (Fischbein and Gazit, 1984; LeCoutre and Durand, 1988; LeCoutre, 1992; Van Dooren et al, 2003).

In many probability problems it is necessary not only to list all the possibilities in the sample space, but also to classify them. This second step, which is usually referred to as `aggregation', can cause many children a great deal of difficulty. For example, if you throw two dice at the same time, there are 36 possible equiprobable outcomes (1,1; 1,2; 1,3 etc.). But, if you record the result in terms of the sum of the two numbers thrown, there are only 11 possible outcomes for the sums, which are two to12, and they are not equiprobable: a total of seven is twice as likely as a total of four, for example, because only three of the 36 possible pairs add up to four, whereas six of them add up to seven. Thus the individual outcomes are equiprobable but the aggregated outcomes are not. This difference causes great difficulty to some children (Abrahamson, 2009), and possible ways to address this would be an interesting question for further research.

The importance of the sample space also raises a general cognitive question, which is fairly obvious, but has never been discussed. To work out the sample space, the child must imagine the future in a particular way, and has to think of all the possible events that could occur in a particular context. There is some research on children's anticipation of particular and highly determined future events, but none on their ability to construct an exhaustive list of alternative, and uncertain, possibilities. Studies of this aspect of thinking about probability are sorely needed.

Quantifying probabilities

Probability is a quantity: it is a quantity based on proportions, and is usually expressed as a decimal number, a percentage or a ratio. The solution to most probability problems rests on the calculation of one or more proportions, but a few can be solved on the basis of simple relations like 'more' or 'larger'.

There is some evidence that even babies in their first year of life form expectations about the relative probability of two different possible events (Teglas et al, 2007; Xu and Garcia, 2008; Xu and Denison, 2009). They are surprised when someone draws mostly red balls from a container that they know to contain many more white than red balls. This reaction to an improbable outcome is evidence that they have some idea of the difference between probable and improbable outcomes. However, this is not evidence that they understand the proportional nature of probability.

Proportional reasoning in general, and not just proportional reasoning about probability, is difficult for young children. In the sphere of probability, this difficulty is most clearly illustrated by tasks in which children have to compare two or more different probabilities. Martignon and Krauss (2009) cite an example of this in a problem given to 15 year-olds: 'Box A contains one white and two black marbles.

Box B contains two white and five black marbles. You have to draw a marble from one of the boxes with your eyes covered. From which box should you draw if you want a white marble?' The solution is not to be found in the absolute numbers of the two colours, but in the proportion of white marbles in each box. A large majority of the 15-year-olds given this problem made the wrong choice. Research by Piaget and Inhelder (1975), Falk et al (1980), Fischbein and Gazit (1984), and Falk and Wilkening (1998), does establish that pupils get better at making proportional calculations of probability as they grow older. However, there is no evidence to support Piaget's view that nearly everyone eventually becomes able to reason about probabilities proportionally. It is possible that many people never manage to do so effectively.

Proportions can be thought of, and calculated, in two ways. One is as a relation of a part to the whole. If a box contains two red and six blue marbles, the whole is all the eight marbles and the proportion of red marbles is 2/8 or 0.25, and of blue marbles 6/8 or 0.75, and this proportion is usually expressed as a fraction or a decimal number. The other is as the relation of one part to another, which is expressed as a ratio. In this example, the ratio of red to blue marbles is 2:6 or 1:3. There is good evidence that children come to understand proportions as ratios (part-part relations) before they understand them as fractions (part-whole relations) (Nunes and Bryant, 1996). This important distinction, however, has never been studied systematically in research on children's understanding of probability. Nonetheless, the reports of children's justification for their correct answers in probability comparisons in Piaget and Inhelder's (1975) and in Fischbein's research suggest that for the most part they used ratios rather than fractions in their reasoning (Fischbein, 1987; Fischbein and Gazit, 1984). The implication of this hypothesis is that children would learn about probabilities more easily if they are initially introduced as ratios.

In many instances, the probability of an event is dependent on the probability of another event. These conditional probabilities often cause adults, as well as children, a great deal of difficulty, as Kahneman and Tversky's (1972) work has established. An example of a conditional probability problem is a question about the likelihood that someone who has tested positive for a particular disease actually has that disease, when the incidence of the disease is 0.001 (or 1 in 1000) and the false positive rate for the test is 0.05 (or 5%). In this case the correct answer is dependent, not just on the false positive rate, but also on the incidence of the disease in the general population. Many people, however, attend only to the false positive rate of the test and not to the incidence of the disease, and this leads them to wildly incorrect calculations (in this example, to the incorrect answer that the probability is 0.95). Recent research has shown that children and adults are much more likely to work out conditional probabilities correctly if the basic information is given as absolute numbers (one person in a thousand has the disease: five out of a hundred people who do not have the disease will test positive) than as decimal fractions (the probability of someone having the disease is 0.001, and the false positive rate is 0.05) (Hoffrage et al, 2002; Zhu and Gigerenzer, 2006). This interesting difference may be connected to the distinction between working with ratios and with fractions in probability problems. It is relatively easy to convert absolute numbers into ratios. Thus, the suggestion of teaching children about probabilities by first presenting these as ratios, rather than as fractions, may hold for conditional, as well as for simple, probabilities. It would be easy to do research on this idea.

Correlations

When two events happen together, their co-incidence might be either a random occurrence or the result of a genuine relationship. Since most such relationships are imperfect (taller people are usually heavier than shorter people but some short people weigh more than expected and some tall people weigh less than expected), we have to work out whether the imperfection of the association amounts to randomness or to a regular relation with exceptions. Thus, correlational thinking depends, at least partly, on an understanding of randomness.

Correlational thinking also depends on children realising that the way to work out whether an association is random or not is to consider the relative amount of confirming and disconfirming evidence. It is difficult to consider the relative amount of confirming and disconfirming evidence without systematic records and their quantification. When people use simple intuitive reasoning, they often fall prey to a confirmation bias: they pay more attention to the confirming than to the disconfirming evidence (Wason, 1968; Evans, 1989; Nickerson, 1998). Examples of this tendency are the idea that someone may have a winning streak in a casino, as if the turning of the roulette wheel had a connection with the player's choice, or that basketball players can have a hot hand, as if the fact that they scored in their last attempt makes it more likely that they will score again (Gilovich et al, 1985). Professionals working in clinical situations must be particularly aware of this confirmation bias: they see a biased sample of people and it is difficult for them to avoid this bias without systematic research (Chapman and Chapman, 1967; 1975). For example, if clinicians think that people only get better from problem *X* with a treatment that they prescribe, they must remember that the people who get better without the treatment are the people that they did not see, so they need to be aware of the risks of confirmation bias.

There is evidence that some adolescents do learn about the need to work out the relationship between the confirming and the disconfirming cases, and to do so proportionally (Inhelder and Piaget, 1958), but it is not clear yet how general this learning is. It is possible that only a minority learn to consider and relate the two kinds of evidence (Adi et al, 1978; Karplus et al, 1980; Batanero et al, 1996), and possibly only in situations where the two types of evidence can be systematically quantified and compared (Ross and Cousins, 1993). If this is the case, education should play a major role in people's understanding of correlation (Vass et al, 2000).

The future of research on children and probability

Research on children's understanding of probability has produced many interesting and educationally valuable conclusions, such as children's understanding of randomness in the context of fairness and the difficulties they have in reasoning proportionally in the context of probability. However, some aspects of children's reasoning about probability have been relatively neglected, such as the cognitive basis for constructing the problem space and the relative effectiveness of presenting and calculating proportions as ratios or as fractions. Another serious gap in research on children's ideas about probability is in longitudinal research, which is needed to establish how well children's early understanding and insights predict their overall learning later on, and also how complete their understanding of probability is by the time they leave school.

We make two main recommendations. The first is to take advantage of research designs that have been successful in research on other aspects of children's intellectual development. In particular, we recommend the combined use of intervention and longitudinal methods to study the links between the four aspects of probability, and to establish what experiences and abilities children need in order to learn about chance and uncertainty. This would provide a scientific basis for the effective teaching of probability.

Our second recommendation is that researchers on children's understanding of probability should pay much more attention to the great amount of related data that exists on other aspects of cognitive development. Probability makes a number of different cognitive demands and most of these demands are shared with other aspects of cognitive development about which we know a great deal. Probability is an intensive quantity, but so are density and temperature. Analyses of the sample space require combinatorial reasoning: so do many branches of scientific thinking. We think that many people doing research on probability have not paid attention to research on these related topics, and have missed out on potentially valuable information.

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These are references made in this summary. A full list of references for the review can be found in the full report, which is available to download from www.nuffieldfoundation.org/probability.

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